Int. J. Heat Mass Transfer. Vol. 7, pp. 1069–1076. Pergamon Press 1964. Printed in Great Britain

TURBULENT HEAT AND MASS TRANSFER IN SMOOTH PIPES†

WILLIAM SQUIRE

Department of Aero-Space Engineering, West Virginia University, Morgantown, West Virginia

(*Received* 8 *November* 1963 *and in revised form* 15 *April* 1964)

Abstract—An analysis of turbulent heat and mass transfer is based on velocity and temperature defect laws in the central portion, combined with a sublayer analysis with velocity as the independent variable. The calculations are in satisfactory agreement with experiments over the entire Prandtl (or Schmidt) number range from 10^{-3} to 10^{3}, except for some discordant results of Frank-Kamenetskii which are generally ignored.

NOMENCLATURE

U, velocity;
u^*, friction velocity;
T, temperature, relative to wall;
t^*, friction temperature $= -\dfrac{\kappa}{u^*}\dfrac{\partial T}{\partial r}\Big|_{r=r_0}$;
r, distance from pipe axis;
r_0, radius of pipe;
A, total Prandtl number [equation (9)];
ν, kinematic viscosity (molecular);
κ, thermal conductivity;
ϵ_m, eddy viscosity (kinematic);
ϵ_h, eddy conductivity for heat;
a, constant in relation for eddy viscosity in sublayer [equation (6)];
X, a dimensionless variable in sublayer analysis [equation (17a)];
$\phi(X)$, a function [equation (17b)];
D, Prandtl constant, 4·07.

Subscripts

av, average;
b. av, bulk average;
δ, value at edge of sublayer.

Superscripts

c, in central core;
S, in sublayer.

Dimensionless groups

Nu, Nusselt number $= C_h\, Re \cdot Pr$;
C_h, heat-transfer coefficient (Stanton number);
C_f, friction coefficient $= (u^*/U_{av})^2$;
Pr, Prandtl number $= \nu/\kappa$;
Pe, Peclet number = Prandtl number × Reynolds number;
Re, Reynolds number;
Re^*, Reynolds number based on friction velocity and eddy viscosity.

INTRODUCTION

IN A PREVIOUS paper [1] the author presented an analysis of turbulent heat and mass transfer in smooth pipes, which was applicable in principle over the entire Prandtl (or Schmidt) number range.‡ However, a simplifying assumption about the laminar sublayer apparently introduced an appreciable error at large Pr, though there was some conflict in the available experimental data [2, 3].

This paper presents an improvement in the analysis based on Spalding's [4] recent treatment of the laminar sublayer using velocity as the independent variable. The previous analysis remains valid in the liquid metal range, but complete agreement is now obtained with Deissler's [2] high Prandtl number results. There is also some improvement in the moderate Prandtl number range.

BASIC ASSUMPTIONS

The present analysis is based on similarity

† This work was supported by NSF Grant 24534.

‡ Both reference 1 and this paper are written in terms of temperature profiles, Prandtl number, and heat transfer. However, the results can be applied to mass transfer by replacing temperature by concentration and Prandtl number and Schmidt number.

assumptions which could, in principle, be tested by experiment. However, at present the available data are inadequate. It should be emphasized that the agreement between the calculated heat-transfer coefficients with experiment is not a satisfactory verification of the assumptions, since many other analyses based on different assumptions also give satisfactory results.

The basic assumptions are that:

(1) there is a central core in which the velocity and temperature distributions obey defect laws, and
(2) there are temperature and velocity sublayers of equal thickness.

There is a reasonable amount of evidence for the defect laws. The velocity defect law is, of course, an established conventional theory, but not the temperature defect law. While the temperature defect law can be supported by dimensional arguments analogous to those for the velocity defect law and experimental results cited in [1] and recent measurements in air by Johnk and Hanratty [8] support this assumption, there are no good measurements in the liquid metal range. As the idea of a temperature defect law leads to temperature profiles at very small Pr which differ markedly from those predicted by conventional theories [9], further studies are in order.

There is no firm experimental basis for the treatment of the sublayer at present, and the assumption of equal thickness of the thermal and velocity sublayer at all Prandtl numbers cannot be considered inherently plausible. Because of the difficulty in making measurements close to a wall, data on the sublayer is very limited. Lin [10] has done some very interesting work with an interferometric technique which showed that the concentration profile began to deviate from a straight line when U/u^* was about 1. However, the measurements were for Reynolds numbers below 13×10^3, and it would be desirable to have similar data over a large range of Re.

ANALYSIS

From the definitions (see Nomenclature)

$$Nu = \frac{Pr\ Re}{(T_{av}/t^*)/(U_{av}/u^*)} = Pr\ Re\ C_h \qquad (1)$$

This is based on an average temperature defined by

$$T = \int_0^1 T\, d(r/r_0)^2 \qquad (2a)$$

rather than on the bulk average defined by

$$T_{b.\,av} = \frac{\int_0^1 U\, T\, d(r/r_0)^2}{\int_0^1 U\, d(r/r_0)^2} \qquad (2b)$$

on which most of the experimental data is based. The correction factor which is evaluated in Appendix A is plotted in Fig. 6. It can be seen that the correction is small, except for small Re and Pr.

It is obvious that

$$\frac{U_{av}}{U^*} = \frac{U_{av} - U_\delta}{u^*} + \frac{U_\delta}{u^*} \qquad (3a)$$

and

$$\frac{T_{av}}{t^*} = \frac{T_{av} - T_\delta}{t^*} + \frac{T_\delta}{t^*}. \qquad (3b)$$

On the basis of the velocity defect law for smooth pipes $(U_{av} - U/u^*)$ is taken as a constant independent of Reynolds or Prandtl number, while (U_δ/u^*) is a function of Reynolds number. By definition

$$\frac{C_f}{2} = \frac{u^*}{U_{av}} = \left[\frac{U_{av} - U_\delta}{u^*} + \frac{U_\delta}{u^*}\right]^{-2}. \qquad (4)$$

In [7], the value of $(U_{av} - U_\delta/u^*)$ was taken as 10·3, and the value of (U_δ/u^*) was determined from experimental C_f values. It was found that

$$\frac{U_\delta}{u^*} = 5{\cdot}25 \log_{10} 10^{-3}\, Re \qquad (5)$$

gave a good fit. The resulting expression,

$$\frac{C_f}{2} = [10{\cdot}3 + 5{\cdot}25 \log_{10} Re \times 10^{-3}]^{-2} \qquad (6)$$

is as good as any of the generally accepted formulae for smooth pipes.

It is assumed that where the defect law is valid, i.e. outside the sublayer, the temperature distribution is related to the velocity distribution by

$$\frac{T_c - T}{t^*} = A(Re, Pr)\frac{U_c - U}{u^*}. \qquad (7)$$

The assumption of similarity of the velocity and temperature defect profiles then gives

$$Nu = \frac{Re\,Pr\,\sqrt{(C_f/2)}}{10{\cdot}3\,A + (T_\delta/t^*)}. \tag{8}$$

The proportionality factor, A, can be interpreted as a total Prandtl number and evaluated from values at the center of the pipe by the expression

$$A = \frac{\nu + \epsilon_m^c}{K + \epsilon_h^c}$$

$$= Pr\,\frac{Re^* + Re\,\sqrt{(C_f/2)}}{Re^* + (\epsilon_h^c/\epsilon_m^c)\,Pr\,Re\,\sqrt{(C_f/2)}}$$

$$= \frac{\epsilon_m^c}{\epsilon_h^c}\,\frac{Re\,\sqrt{(C_f/2)} + Re^*}{Re\,\sqrt{(C_f/2)} + (\epsilon_m^c\,Re^*/\epsilon_h^c\,Pr)} \tag{9}$$

where $Re^* = (2r_0u^*/\epsilon_m^c)$ should be between 40–50 [11].

It can be seen that unless Pr is very small, A is close to $\epsilon_m^c/\epsilon_h^c$ and that the exact value of Re^* will only be important in the liquid metal range.

In [1] it was assumed that

$$\frac{T_\delta}{t^*} = Pr\,\frac{U_\delta}{u^*}. \tag{10}$$

This assumption is equivalent to neglecting turbulent transport in the sublayer. For large Pr where $10{\cdot}3\,A$ can be neglected, it results in

$$Nu = \frac{Re\,\sqrt{(C_f/2)}}{1 - 10{\cdot}3\,\sqrt{(C_f/2)}} \tag{11}$$

which is independent of Pr. This contradicts most of the available experimental data which has been summarized by Deissler [2] but is in agreement with the views and data of Frank-Kamenetskii [3].

The present analysis of the laminar sublayer is based on Spalding's recent idea of using the velocity as the independent variable. Because the sublayer is thin, the relation

$$\frac{d(T/t^*)}{d(U/u^*)} = \frac{\nu + \epsilon_m^s}{K + \epsilon_h^s} \tag{12}$$

can be used. Spalding suggests

$$\frac{\epsilon_m^s}{\nu} = \left(\alpha\,\frac{U}{u^*}\right)^4 \tag{13}$$

and for simplicity it is assumed that

$$\epsilon_m^s = \epsilon_h^s. \tag{14}$$

Then integration of (12) gives

$$\frac{T_\delta}{t^*} = Pr\int_0^{U_\delta/u^*} \frac{1 + [\alpha\,(U/u^*)]^4}{1 + Pr\,[\alpha\,(U/u^*)]^4}\,d\frac{U}{u^*}. \tag{15}$$†

This integral can be evaluated by standard methods to give

$$\frac{T_\delta}{t^*} = \frac{U_\delta}{u^*}\,[1 + (Pr - 1)\,\phi(X)] \tag{16}$$

where

$$X = \alpha\,Pr^{1/4}\,\frac{U_\delta}{u^*} = 2/3\,Pr^{1/4}\log_{10} Re \times 10^{-3} \tag{17a}$$

and

$$\phi(X) = \frac{1}{x\,\sqrt{(32)}}\left|\ln\frac{x^2 + x\,\sqrt{(2)} + 1}{x^2 - x\,\sqrt{(2)} + 1} + 2\tan^{-1}\frac{2x}{1 - x^2}\right|. \tag{17b}$$

Figure 1 shows $\phi(X)$ as a function of X. It can be seen that for $X < 1/2$, equation (10) is a valid approximation. Also, for Pr close to 1, equation (10) will be accurate even though X is large. On the other hand, for $X > 3$,

$$\phi(X) \simeq \frac{2\pi}{X\,(\sqrt{32})}.$$

Figure 2 shows lines of constant X in the Re–Pr plane.

The final expression, therefore, is

$$Nu = \frac{Re\,Pr}{[10{\cdot}3 + (U_\delta/u^*)] + \{10{\cdot}3\,A + (U_\delta/u^*)\,[1 + (Pr - 1)\,\phi(X)]\}} \tag{18}$$

where

U_δ/u^* is given by equation (5)
A is given by equation (9)
$\phi(X)$ is given by equation (17).

† It is at this point that the assumption of equal thicknesses enters.

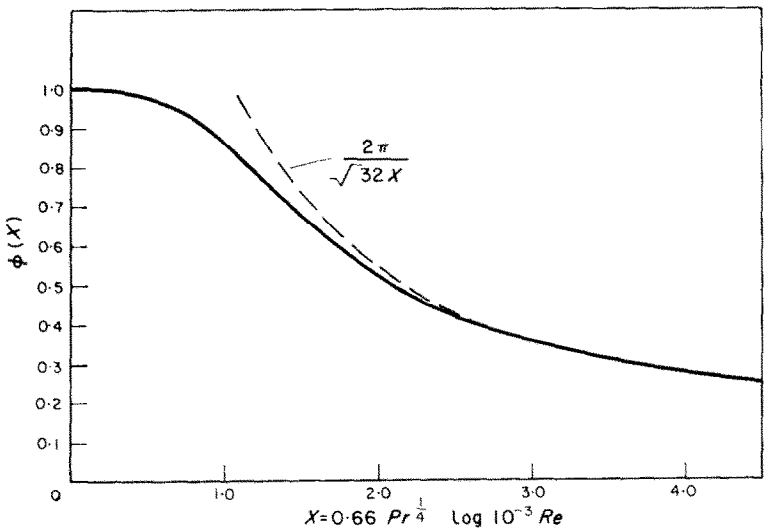

FIG. 1. Plot of $\phi(X)$ and approximate expression for large X.

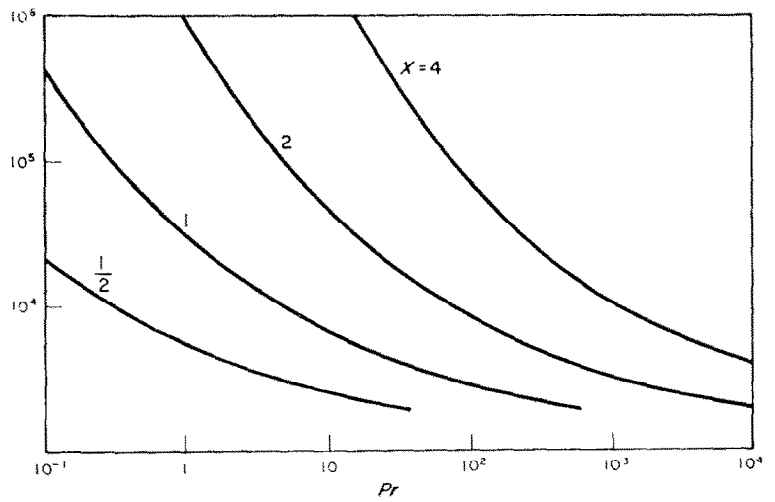

FIG. 2. Contours of equal X in Reynolds number–Prandtl number plane.

SPECIFIC RESULTS

Equation (18) contains three constants: Re^*, $(\epsilon_m^c/\epsilon_h^c)$ and α which have to be evaluated. Fortunately, there are three regions in which a different constant predominates. For very large Pr only α is important, at $Pr \simeq 1$, it is $\epsilon_m^c/\epsilon_h^c$ which dominates, and in the liquid metal range Re^* controls.

For large Pr, equation (18) can be approximated by

$$Nu = \frac{Re\ Pr\ \sqrt{(C_f/2)}}{Pr\ (U_\delta/u^*)\ \phi(X)} \tag{19a}$$

and since X is also large, this becomes

$$Nu = \frac{2\alpha}{\pi}\ Re\ Pr^{1/4}\ \sqrt{(C_f)} \tag{19b}$$

This is exactly the result obtained by Deissler (equation (32) of [2]), so a can be taken as 1/8. While it was to be expected that the present analysis would give the same functional dependence of Nu on Re, Pr and C_f, the identity of the proportionality constant is noteworthy. Because of this identity, the agreement with the experimental data (excluding Frank-Kamenetskii's) at large Pr will be very good, as shown by Fig. 4 of reference 2.

It appeared originally that they theory could account for Frank-Kamenetskii's results as a Reynolds number effect, since his measurements were made for $Re < 10^4$. However, his results differ markedly from the present analysis and all the generally accepted theoretical and empirical expression, so that there must be some unexplained error involved.

For $Pr = 1$, the effect of a cancels out and Re^* has a very minor effect, In [1] the value of $(\epsilon_m^c/\epsilon_h^c)$ was taken as 0·85. This value was based on the value of $(C_f/2C_h)$ at $Re = 1{\cdot}5 \times 10^5$ and $Pr = 0{\cdot}72$ of 0·85 cited by Hastrup [13]. According to the analysis used in [1], this ratio did not change appreciably with Pr between 0·72 and 1·0. With the present analysis taking $A = 1{\cdot}0$ gives good agreement with both Hastrup's value and more recent experimental results in this range reported by Dipprey and Sabersky [13], as shown in Fig. 3. Furthermore, with this value A also approaches 1 in this range, which is in better agreement with the experimental temperature traverses.

In the liquid metal range X is small so the analysis of [1] is valid. However Nu was calculated using the complete new expressions. The value of $a = 1/8$ and $\epsilon_m^c/\epsilon_h^c = 1$ were used, though these values have little effect in the liquid metal range. The value of Re^* was

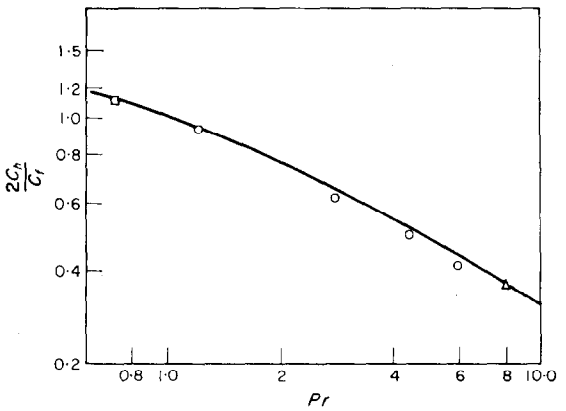

FIG. 3. Comparison of theory and experiment for moderate Prandtl numbers at a Reynolds number of $1{\cdot}5 \times 10^5$. Experimental values from [13].

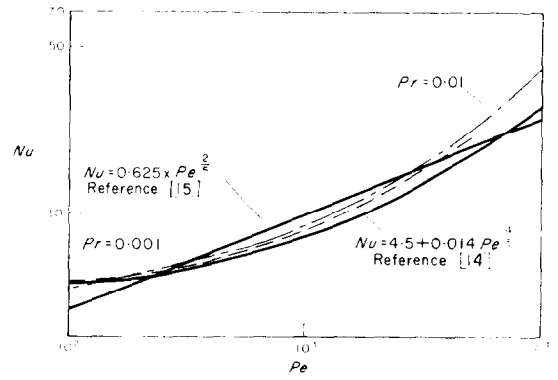

FIG. 4. Comparison of theory and empirical expressions for the liquid metal range.

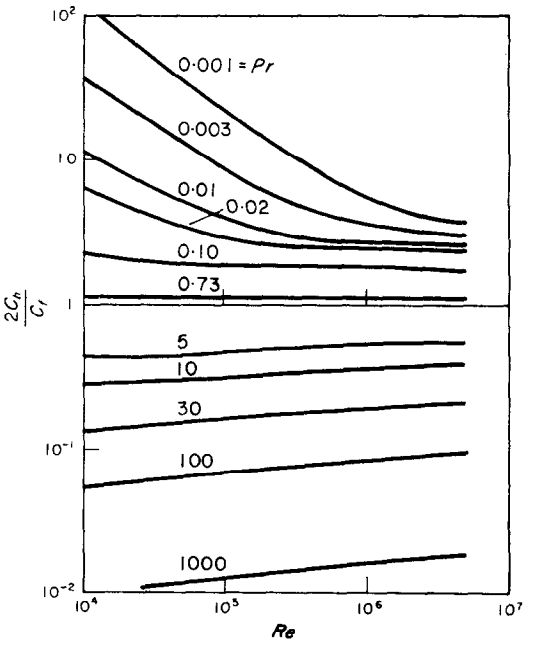

FIG. 5. Plot of $2C_h/C_f$ versus Reynolds number for different Prandtl numbers.

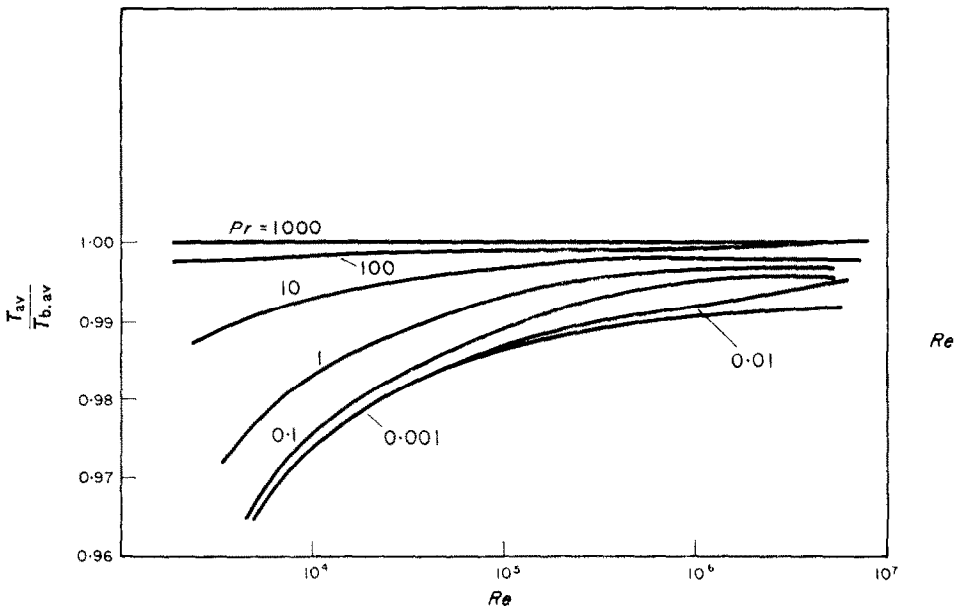

FIG. 6. Plot of correction factor ($T_{av}/T_{b.av}$) vs Reynolds number for different Prandtl numbers.

taken as 50. It is well known that in the liquid metal range to a high degree of precision Nu is a function of the Peclet number ($= Re \times Pr$). The plot of our results in Fig. 4 shows that this is the case. The empirical relations

$$Nu = 4{\cdot}5 + 0{\cdot}014\, Pe^{4/5} \qquad (20a)$$

due to Mikheev [14] and

$$Nu = 0{\cdot}625\, Pe^{2/5} \qquad (20b)$$

due to Lubarsky and Kaufman [15] are shown for comparison. These empirical expressions have been used instead of the experimental data, because of the large amount of scatter in the data and because Mikheev's data, which is the largest consistent set, has only been published as a small curve.

In Fig. 5, the values of $2(C_h/C_f)$ over the entire range of Re and Pr are shown. From this and Fig. 1, values of Nu or C_h can be obtained. By using Fig. 6, the result can be corrected to a bulk average temperature.

CONCLUSIONS AND PLANS FOR FUTURE WORK

The present analysis gives heat- (and mass-) transfer coefficients in reasonably good agreement with experiment over the entire Prandtl (and Schmidt) number range. As there are numerous analyses which will work for part of the range, and two other analyses [5, 6] which also cover the entire range of Prandtl numbers, this is not a revolutionary achievement. However, the present analysis appears much simpler than those commonly used, because it does not involve the evaluation of velocity gradients, i.e. it is an integral rather than a differential approach.

This paper has dealt with the constant flux fully developed case for a fluid with constant properties flowing in a smooth circular pipe. In future work, it is hoped to remove these restrictions and deal with more general problems. In particular, the problem of the effect of roughness is of considerable interest at present. Since the defect law is applicable to the velocity distribution in rough pipes, only a modification of the sublayer analysis is needed to make the analysis applicable.

APPENDIX A

An approximate calculation of $T_{av}/T_{b.av}$ is

presented, based on the defect laws and neglecting the sublayer. By definition

$$\frac{T_{av}}{T_{b.\,av}} = \frac{\int_0^1 T\,d\eta \; x \int_0^1 U\,d\eta}{\int_0^1 T\,U\,d\eta} \tag{A1}$$

where $\eta = (r/r_0)^2$. This can be rewritten as

$$\frac{T_{av}}{T_{b.\,av}} = \frac{\int_0^1 \frac{T_c}{t^*} - \frac{T_c - T}{t^*}\,d\eta \; x \int_0^1 \left(\frac{U_c}{u^*} - \frac{U_c - U}{u^*}\right) d\eta}{\int_0^1 \left(\frac{T_c}{t^*} - \frac{T_c - T}{t^*}\right)\left(\frac{U_c}{u^*} - \frac{U_c - U}{u^*}\right) d\eta} \tag{A2}$$

By assuming

$$\frac{U_c - U}{u^*} = f(\eta) \tag{A3a}$$

$$\frac{T_c - T}{t^*} = A f(\eta) \tag{A3b}$$

over the entire cross section equation (A2) becomes

$$\frac{T_{av}}{T_{b.\,av}} = \frac{\frac{T_c}{t^*}\frac{U_c}{u^*} - \left(A\frac{U_c}{u^*} + \frac{T_c}{t^*}\right)\int_0^1 f\,d\eta + A\left(\int_0^1 f\,d\eta\right)^2}{\frac{T_c}{t^*}\frac{U_c}{u^*} - \left(A\frac{U_c}{u^*} + \frac{T_c}{t^*}\right)\int_0^1 f\,d\eta + A\int_0^1 f^2\,d\eta}. \tag{A4}$$

By Prandtl's rule

$$\int_0^1 f\,d\eta = \frac{U_c - U_{av}}{u^*} = D = 4{\cdot}07 \tag{A5}$$

In order to evaluate $\int_0^1 f^2\,d\eta$ an approximation

$$\frac{U_c - U}{u^*} = 3D\,[1 - \sqrt{(1-\eta)}] \tag{A6}$$

derived in [7] is used. This gives

$$\int_0^1 f^2\,d\eta = 3/2\;D^2 \tag{A7}$$

Therefore, the value of $(T_{av}/T_{b.\,av})$ can be computed as a function of Re and Pr by using the expressions for A, U_δ/u^*, and (T_δ/t^*) given in the paper. The results are shown in Fig. 6.

REFERENCES

1. W. Squire, An extended Reynolds analogy, *Proceedings Sixth Midwestern Conference on Fluid Mechanics*, p. 16. University of Texas, Austin (1959).
2. R. G. Deissler, Analysis of turbulent heat transfer, mass transfer, and friction in smooth tubes at high Prandtl and Schmidt numbers, *NACA* Report 1210 (1955).
3. D. A. Frank-Kamenetskii, Diffusion and heat exchange in chemical kinetics, pp. 68, 195. Princeton University Press, Princeton (1955).
4. D. B. Spalding, New analytical expression for drag of flat plates valid for both turbulent and laminar regimes, *Int. J. Heat Mass Transfer* **5**, 1133–1138 (1962).
5. H. W. Kropholler and A. D. Carr, The prediction of heat and mass transfer coefficients for turbulent flow in pipes at all values of the Prandtl or Schmidt numbers, *Int. J. Heat Mass Transfer* **5**, 1191–1205 (1962).
6. V. G. Levich, *Physicochemical Hydrodynamics*, p. 204. Prentice Hall, Englewood Cliffs (1962).
7. W. Squire, A unified theory of turbulent flow III. Pipe flow, *Appl. Sci. Res.* **A10** 23–44 (1961).
8. R. E. Johnk and T. J. Hanratty, Temperature profiles for turbulent flow of air in pipe, *Chem. Engng Sci.* **17**, 867–879 (1962).
9. W. Squire, Reply to comments by Rogers and Mayhew, *Int. J. Heat Mass Transfer* **5**, 559–560 (1962).
10. C. S. Lin, R. W. Moulton and G. L. Putnam, Mass transfer between solid wall and fluid streamers, *Ind. Engng Chem.* **45**, 636–645 (1953).
11. I. Tani, Theory of bluff bodies, University of Maryland Technical Note *BN*-104 (1957).
12. R. C. Hastrup, R. H. Sabersky, D. R. Bartz and N. B. Noel, Friction and heat transfer in a rough tube at varying Prandtl numbers, *Jet Propuls.* **28**, 259–263 (1958).
13. D. F. Dipprey and R. H. Sabersky, Heat and momentum transfer in smooth and rough tubes at various Prandtl modulus, *Int. J. Heat Mass Transfer* **6**, 329–353 (1963).
14. M. A. Mikheev, V. A. Baum, K. D. Vosdresensky and O. S. Feydynsky, Heat transfer by molten metals, *Proc. Int. Conf. Peaceful Uses Atomic Energy* **9**, 285–289, United Nations, New York (1956).
15. B. Lubarsky and S. J. Kaufman, Review of experimental investigations of liquid-metal heat transfer, *NACA TN* 3336 (1955).

Résumé—Une analyse du transport turbulent de chaleur et de masse est basée sur les lois de différences de vitesse et de température dans la partie centrale et combinée avec une analyse de la sous-couche limite où la vitesse est prise comme variable indépendante. Les calcues sont en accord satisfaisant avec les expériences sur toute la gamme de nombres de Prandtl (ou de Schmidt) allant de 10^{-3} à 10^3, sauf pour quelques résultats discordants de Frank-Kamenetskii qui sont généralement ignorés.

Zusammenfassung—Die Analysis des turbulenten Wärme- und Stofftransports wird auf Gesetze der Geschwindigkeits- und Temperaturdefekts im mittleren Abschnitt zurückgeführt in Verbindung mit einer Unterschichtanalysis, wobei die Geschwindigkeit als unabhängige Variable dient. Die Berechnungen zeigen zufriedenstellende Übereinstimmung mit Versuchen im gesamten Bereich der Prandtl- (oder Schmidt-) Zahl von 10^{-3} bis 10^3, ausser einigen abweichenden Ergebnissen nach Frank-Kamenetskii, die gewöhnlich unberücksichtigt bleiben.

Аннотация—Проводится анализ турбулентного тепло-и массопереноса на основе законов недостатков скорости и температуры как в ядре течения, так и в подслое, где в качестве независимой переменной используется величина скорости. Расчеты хорошо согласуются с результатами опытов во всем диапазоне чисел Прандтля (или Шмидта) от 10^{-3} до 10^3. Имеется некоторое расхождение с результатами Франка-Каменецкого.